Element is relation: A postscript to the "Order without orientation"

Group theory is known to have an interesting property called Cayley's theorem.

> **Cayley's theorem:** Every group is isomorphic to a permutation group. In particular:
>
> For each element a of G, let f_a be a function f_a:G→G such that $f_a(x) = a\cdot x$. Each such function is a permutation from G to G. Let G' be the set of all such permutations. And, let ∘ be the function composition. Then, <G', ∘> forms a permutation group such that it and <G, ·> are isomorphic with the isomorphism f: G'→ G such that $f(f_a) = a$.

Ying theory exhibits a similar property. For the ease of discourse, let me call it Mobiusness theorem.

> Theorem 22
> **Mobiusness theorem:** Let γ be an equivalence relation on Y^2 such that <a,b> γ <c,d> iff a:b = c:d. Let :: be a binary operation on Y^2/γ such that [a,b] :: [c,d] = [(a:b),(c:d)]. ([a,b] denotes the γ-equivalence class generated from <a,b>.) <Y^2/γ, ::> forms a ying such that it and <Y, :> are isomorphic with the isomorphism f: Y^2/γ →Y such that f[a,b]=a:b.

Because of these properties, in each theory, a strange sort of identity or indiscernibility is obtained between elements of a model and a certain kind of binary relation on it. (A permutation f_a:G→G is a binary relation on G, and so is an equivalence class [a,b]∈ Y^2/γ.) Once you realize the total structure of a model, even the distinction between an element and a (certain type of binary) relation loses its absoluteness or inherentness, and becomes looking like an indexical distinction of some sort. Locally, the distinction appears to be absolute, but globally, it appears to be relative, in particular, relative to "us," our arbitrary fixed local distinction. It is because of this that I initially named this property of the ying theory Mobiusness (although the analogy is obviously not perfect), long time ago when I knew nothing about group theory or Cayley's theorem.[1]

This similarity temps us to wonder whether Cayley's theorem and Mobiusness theorem are equivalent for any binary operational theory, that is, whether, for any theory Γ of the following type, Γ admits of its version of Cayley's theorem iff it admits of its version of Mobiusness theorem.

> **A binary operational theory:** Let Γ be a theory whose definition makes use of no other non-logical notions than a binary operation · and the equality =. (The definition contains no predicate except the equality and no individual constant.) We call such Γ a binary operational theory.

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[1] I first discovered that both <**Z**, **-**> and <$\mathbf{Q}^+$, ÷> exhibited Mobiusness, and realized intuitively that they were "form-wise identical." This was sometime from 2001 to 2003, when I was studying logic and mathematics by myself. I studied abstract algebra and learned about the Cayley's theorem sometime in 2006. My memory cannot be trusted, but I think that the idea of axiomatizing the theory of which these two structures are models somehow never occurred to me before, and the current attempt ("Order without orientation") is my first attempt to axiomatize the theory.

At least, group theory admits its version of Mobius theorem. However, Ying theory proves not to admit of its version of Cayley's theorem. <Y', ∘> is not a ying because $f_a \circ f_e \neq f_a$. (Note: f_e is not the identity permutation but the e-inverse permutation.)

However, Cayley's theorem may be modified as follows:

> **Modified Cayley theorem:** For each element a of G, let f_a be a function f_a:G→G such that $f_a(x) = a \cdot x$. Each such function is a permutation from G to G. Let G' be the set of all such permutations. And, let ∘ be a binary operation on G' such that $f_a \circ f_b = f_{(a \cdot b)}$. <G', ∘>, which is no longer a permutation group, still forms a group such that it and <G, ·> are isomorphic with the isomorphism f: G'→ G such that $f(f_a) = a$.

Ying theory admits its version of modified Cayley theorem. And, this modification of Cayley theorem leaves intact the original theorem's similarity to Mobiusness theorem. I do wonder whether this modified Cayley's theorem and Mobiusness theorem are equivalent for any binary operational theory.

I also wonder how, in a binary operational theory, the following theorem is related to modified Cayley theorem and Mobiusness theorem. (The following is in a sense a theorem-schema, in the sense that it is not described as a theorem proven of any definite theory, but as a possible theorem that any theory with a binary operation may admits.)

> **Bidirectional eliminability theorem:** (We assume that we are now talking about a hypothetical theory Γ with a binary operation ·.) $a \cdot x = a \cdot y \Rightarrow x=y$ and $x \cdot a = y \cdot a \Rightarrow x=y$. If Γ admits this theorem, we say that the binary operation · is bidirectionally eliminable.

In particular, I wonder if admitting this theorem is equivalent, in a binary operational theory, of admitting its version of modified Cayley theorem and Mobiusness theorem. At least, both group theory and ying theory admit their version of this theorem.

One thing is obvious to me, as of now. If the binary operation · of any theory Γ is bidirectionally eliminable, the table that describes the behavior of · (Γ's version of group table) would be a smallest exhaustive permutations table.

> **Smallest exhaustive permutations table:** Let M be a set $\{a_1, a_2, a_3, \ldots\}$. (M may or may not be finite.) Let X be a set of permutations f from M to M. We can represent X by a table such that each row represents a permutation f∈X such that $f(a_i)$ is given in the $_i$-th place of f's row. We call such a table X's permutations table. X's permutations table is *exhaustive* if each element a_i of M appears in each of the $_j$-th place of the permutation at least in some permutation f in X. The table is a smallest exhaustive permutations table if each element a_i of M appears in each of the $_j$-th place exactly once.

It seems to me that for a binary operational theory Γ, it may be necessary and sufficient for Γ to admit its version of modified Cayley theorem and Mobiusness theorem that the Γ table is a smallest exhaustive permutations table.

But I need proof.