On the nonemptiness requirement and two interpretations of quantifiers

Students of abstract mathematics often encounter in their textbooks an axiomatic definition of an abstract mathematical concept in which the so-called "underlying set" is explicitly required, as a part of the definition, to be nonempty. So, for example, the concept of group is often defined as a pair <G, ·> of a nonempty set G and a binary operation ·, which satisfies certain conditions (axioms). Similarly, the concept of Boolean algebra is often defined as an ordered six-tuple <B, ∧, ∨, 1, 0, ¬> which satisfies certain conditions, where the "underlying set" B has at least one (or two) elements. I suspect that the main reason for which mathematicians make this sort of explicit nonemptiness requirement for some concepts is that the concept to be defined is *existentializable* in the following sense: an abstract mathematical concept is *existentializable* if an existential theorem can be derived from some of its alternative axiomatic definitions. Needless to say, an existential theorem can be derived from some of the alternative definitions if and only if it can be derived from *all* the alternative definitions. So, existentializability is a property of an abstract concept rather than of an axiomatization of it.[1] Obviously, the concepts of group and Boolean algebra are both existentializable. The standard axiomatization of the concept of group includes an existential axiom, which is *a fortiori* an existential theorem of the definition. The standard six-tuple definition of the concept of Boolean algebra has no existential axiom, but, an existential theorem easily follows from it *via* an inference-pattern which is called "existential generalization" by some logicians because the definition uses *constant* symbols (namely, "1" and "0" above).

Here, by a "constant symbol," I mean to draw a contrast to a *variable* symbol, of which it is characteristic and essential that it is used, in an actual mathematical reasoning or discourse (as opposed to a "formalized" one), *only* in combination with quantifiers. That is, I mean by a "constant symbol" a mathematical equivalent of a "proper name" in ordinary linguistic contexts --- equivalent in the sense that they are both *directly referential* symbols, such that any *use* of them (as opposed to *mention* of them) makes the users *ontologically committed* to the "existence" in respective senses of the referents of the symbols *directly*, that is, without relying on existential quantification.

Digression: As is well-known, this pragmatic phenomenon accompanying the use of a directly referential symbol has caused many philosophical puzzles which characterize a good part of the post-Fregean philosophy of language in the analytic tradition, not in the least by way of Russell's theory of description. In my diagnosis, however, this theory is based on a certain static and thus unrealistic view of language, which is nonetheless "natural" in the sense that it is, in my hypothesis, built in any natural human language just as the so-called "natural theory of mind" is. So,

[1] By the way, the naming of this property is based on my wild guess, not yet proved, that if a concept is existentializable in this sense, it has some alternative axiomatization which makes no use of a constant symbol but has some existential axiom. So, the naming is roughly to signify that the concept so described can be defined without a constant symbol but with an existential axiom.

if the "natural theory of mind" can be called a *folk psychological* theory, then, by the same token, we can say that the static view of language on which Russell's theory is based is also a *folk linguistic* theory. In my opinion, a good part of the post-Fregean analytic philosophy not just of language but also of mathematics, of knowledge (epistemology), of logic, of mind, --- in general, the post-Fregean analytic philosophy as a whole --- has been largely "trapped," as it were, in a folk linguistic preconception as to what language is and is for, or what it is that we do with language. Though this preconception is not at all useless or worthless, an alternative conception seems to have been increasingly anticipated recently. (If you are interested in what sort of alternative I'm thinking about, then, please see this if you can read Japanese, or this if you cannot.) ----- **Digression ended**

Now, I think that the following (1) is probably accepted by most mathematicians due to their intuitive interpretation of existential quantification, which I think is a part of the aforementioned folk theory of language:

> **(1) If an abstract concept is existentializable, then, it cannot have any empty model.**

This is because the "underlying set" of a definition of an existentializable concept cannot be interpreted to be empty because an existential theorem requires that there be, in the "underlying set," an element which satisfies the property described by the existential theorem. So, at least for some people, it may be that when they explicitly make the nonemptiness requirement in axiomatically defining some existentializable concept, it is only to make explicit what (they think) is already implicitly contained in the concept itself.

However, the concepts of group and Boolean algebra are also *universalizable* in the following sense: An abstract concept is *universalizable* if it can be axiomatically defined (i) without an existential axiom *and* (ii) without a constant symbol.[2] As regards universalizable concepts, I think we (as working mathematicians) accept the following (2) by our intuitive interpretation of universal quantification --- with some residual uneasiness, perhaps:

> **(2) If an abstract concept is universalizable, then, the empty set *vacuously* constitutes (the "underlying set" of) an empty model.**

After all, an axiom-set which consists exclusively of universal axioms is in fact a single universal statement; and, it does fit certain mathematical or logical intuition, and is a common mathematical practice as well, to take the universal quantification without the so-called existential import (at least in certain contexts --- see below). If I'm not alone, our intuitive rejection of existential import in interpreting the universal quantification is not without a lingering sense of counter-intuitiveness when judged by our "natural" intuition about the universal quantification. However, we learn to

[2] See this planetmath article, e.g., for such a universalized axiomatization of group. See Robert Stoll's *Set Theory and Logic* (in particular, Theorem 3.1 of Chapter 6, in p. 255 in the Dover edition, which is the 272nd page in the PDF ebook linked here) for that of Boolean algebra.

suppress this sense of counter-intuitiveness the more we become competent as working mathematicians (i.e., the more we become competent in "mathematese").

Back to the nonemptiness requirement in definitions of group and Boolean algebra: Obviously, the properties of existentializability and universalizability are not mutually incompatible in the same concept, as clearly exemplified by the concepts of group and Boolean algebra. So, the two intuitions (1) and (2) collide when it comes to those concepts. My guess is that the current consensus among mathematicians about this collision is to choose to adopt the intuition (1), at the cost of rejecting the intuition (2). So, when it comes to a concept that is both existentializable *and* universalizable, it may be that mathematicians make the explicit nonemptiness requirement in its definition to signify this nontrivial choice, as a consensus of the mathematical community.

However, I think that there is another solution for this collision of intuitions. We can interpret our own (past) mathematical discourses and/or reasonings such that (i) we always have had, tacitly, two distinct systematic interpretations of the quantifiers, and (ii) we have been using them in an *erroneously* mixed way, until this day. --- Here, I'm implying that there is a *correct* way of using these distinct interpretations in mixture, in mathematics. I will come back to this soon. --- Of these two systematic interpretations, the one may be called the *descriptive-semanticist* interpretation and the other the *normative-pragmaticist* interpretation of the quantifiers.[3]

According to the descriptive-semanticist interpretation, both the universal and existential quantifiers are interpreted "naturally," so that neither existential statements nor universal statements can be rendered true by an empty "underlying set." (An empty "underlying set" renders universal statements meaningless or irrelevant and existential statements false.) So, according to this interpretation, it is just as it should be that the universal quantification is interpreted with existential import. By contrast, according to the normative-pragmaticist interpretation, neither existential statements nor universal statements are to be interpreted with existential import. Just as mathematicians customarily take the universal quantification without existential import (if they encounter one in certain kinds of context), existential quantification too can be easily taken in the same way. An existential statement of the form $\exists xFx$ is only interpreted, in that case, to stipulate that $\neg\forall x\neg Fx$, that is, "Not all x (if there are any) are not-F." Perhaps, when working in or with the normative-pragmaticist interpretation, the existential quantifier may be rather called the *exceptive stipulator*. Similarly, the universal quantifier may rather be called the *categorical stipulator*. (As such, the "quantifiers" or stipulators are auxiliary vocabularies with which we mark or signify

[3] Here, the terminology "pragmaticist" primarily implies that it is a derivation from the word "pragmatics." In fact, the whole expression, "normative-pragmaticist," pays homage to Robert Brandom's notion of normative pragmatics. However, it is also a derivation from Peirce's "pragmaticism" as well, for, I think Brandom's normative pragmatics, with its inseparable connection with his inferential semantics, is an extension of pragmatism as a philosophical doctrine Peirce introduced.

which of two sub-kinds of the (sub-locutionary) illocution of quantification we are performing. I cannot explain what I mean by this in full detail yet. I'm in the middle of writing a revision of my March 5, 2015 post of this "blog," in which I will explain it in more details.)

Above, I stated that there is a correct way of mixing these two interpretations in mathematics. Roughly speaking, in my opinion, we are to use the normative-pragmaticist interpretation when we are encountering "quantifications" in *inferential* contexts. We are to use the descriptive-semanticist interpretation when we are encountering them in *referential* contexts. However, when we draw an inference, we resort to some referential discourse (or reasoning) by way of which we somehow examine whether the inference is valid or not. Hence, in mathematics, *locally referential* contexts permeate, ubiquitously, "inside" of *globally inferential* contexts. I cannot explain the details of this yet, either. (Please wait for the aforementioned revision of March 5, 2015 post for this too.) One thing I can and should state here is that, according to this opinion, an axiomatic *definition* of an abstract mathematical concept constitutes an inferential context, and so does a theorem derived from such a definition. So, when we encounter "universal and existential quantifiers" in such a context, we should understand them to be categorical and exceptive stipulators, respectively.

Hence --- coming back to the intuition-collision with respect to existentializable and universalizable concepts ---, I argue that we should accept the intuition (2), in rejection of (1), which is not at all a *cost* because an "existentializable" concept carries no ontological commitment because an "existential" theorem is actually an exceptive theorem. So, we do not have to make, or rather, we should not make, the nonemptiness requirement in defining any abstract concept, "universalizable" or not, "existentializable" or not.